RESEARCH ARTICLE OPEN ACCESS

Design and Simulation of Different Controllers for Stabilizing Inverted Pendulum System

Rahul Kr. Rai[1], Rekha[2], A.K.Singh[3]
[1]PG Student, Electrical Engg. Department, NIT Jamshedpur
[2,3]Faculty, Electrical Engineering Department, NIT Jamshedpur

ABSTRACT
The Inverted Pendulum system has been identified for implementing controllers as it is an inherently unstable system having nonlinear dynamics. The system has fewer control inputs than degrees of freedom which makes it fall under the class of under-actuated systems. It makes the control task more challenging making the inverted pendulum system a classical benchmark for the design, testing, evaluating and comparing.
The inverted pendulum to be discussed in this paper is an inverted pendulum mounted on a motor driven cart. The aim is to stabilize the system such that the position of the cart on the track is controlled quickly and accurately so that the pendulum is always erected in its vertical position. In this paper the linearized model was obtained by Jacobian matrix method.
The Matlab-Simulink models have been developed for simulation for optimal control design of nonlinear inverted pendulum-cart dynamic system using different control methods. The methods discussed in this paper are a double Proportional-Integral-Derivative (PID) control method, a modern Linear Quadratic Regulator (LQR) control method and a combination of PID and Linear Quadratic Regulator (LQR) control methods. The dynamic and steady state performance are investigated and compared for the above controllers.

Keywords- Inverted pendulum (IP), nonlinear system, double PID control, optimal control, LQR

I. Introduction

The Inverted pendulum (IP) is an unstable, non-linear multi-variable system which can be treated as a typical control problem to study various modern control theories. The IP belongs to the class of under-actuated systems having fewer control inputs than degrees of freedom. This makes its control task more challenging that makes this system as a classical benchmark for the design, testing, evaluating and comparing of different classical controlling techniques. The IP is among the most difficult systems as it is an inherently unstable system. It is taken as one of the most important classical problems and so the control of inverted pendulum has been a research interest in the field of control system engineering.

Inverted Pendulum provides a good platform for control engineers to verify and apply different logics in the field of control theory. The control of inverted pendulum (IP) resembles the control system that exists in most of the real time applications such as altitude control of space satellites, missiles and rockets, landing of aircrafts, balancing of ship against tide, Seismometer etc.

Due to the above and many other features it is an important choice to analyze its dynamic model and propose a control law. An Inverted Pendulum has its mass above the pivoted point, which is mounted on a cart which can be moved horizontally. The pendulum is stable while hanging downwards, but the inverted pendulum is inherently unstable and need to be balanced. In this case the system has one input - the force applied to the cart, and two outputs - position of the cart and the angle of the pendulum, making it as a single input-multi output (SIMO) system. The aim of this paper is to stabilize the Inverted Pendulum (IP) such that the position of the cart on the track is controlled quickly and accurately so that the pendulum is always erected in its inverted position during such movements. Realistically, this simple mechanical system is representative of a class of altitude control problems whose goal is to maintain the desired vertically oriented position at all times [1-3].

II. Mathematical Modeling of IP System

A. Inverted Pendulum System Equations

The free body diagram of an inverted pendulum mounted on a motor driven cart is shown in Fig. 1. It is assumed here that the pendulum rod is mass-less, and the hinge is frictionless. The cart mass and the ball point mass at the upper end of the inverted pendulum are denoted as M and m, respectively. There is an externally x - directed force on the cart, $x(t)$and a gravity force acts on the point mass at all times.

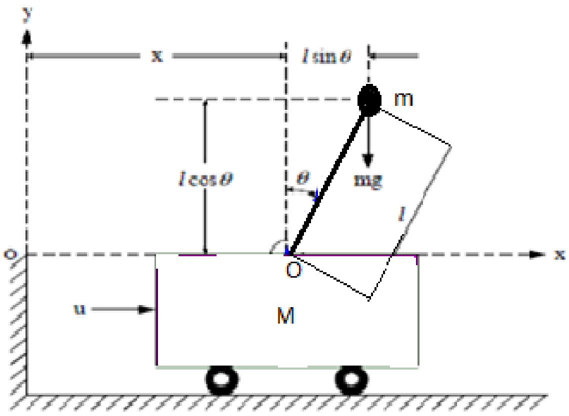

Figure 1: Inverted pendulum mounted on a motor driven cart

The force balance equation along x-plane can be written as:

$$M\frac{d^2}{dt^2}x + m\frac{d^2}{dt^2}x_G = u \qquad (1)$$

Where the timedependent centre of gravity (COG) of the point mass is given by the co-ordinates, $(x_{G,}y_G)$.

$$x_G = x + lsin\theta \text{ and } y_G = y + lcos\theta \qquad (2)$$

$$M\frac{d^2}{dt^2}x + m\frac{d^2}{dt^2}(x + l\,\text{l}\sin\theta) = u \qquad (3)$$

Which gives

$$(M+m)\ddot{x} - ml\sin\theta\,)\dot{\theta}^2 + (ml\cos\theta\,)\,\ddot{\theta} = \text{u} \qquad (4)$$

The resultant torque balance can be written as

$$:(F_X\cos\theta)l - F_y\sin\theta)l = (mg\sin\theta)l \qquad (5)$$

Where the force components, F_x and F_y , are determined as

$$F_x = \text{m}\frac{d^2}{dt^2}x_G = \text{m}[\ddot{x} - (l\sin\theta)\,\dot{\theta}^2 + (l\cos\theta)\ddot{\theta}] \quad \text{and}$$

$$F_y = \text{m}\frac{d^2}{dt^2}y_G = \text{-m}[(l\cos\theta)\,\dot{\theta}^2 + (l\sin\theta)\,\ddot{\theta}] \qquad (6)$$

From equations (5) & (6) , we have

$$\text{or, } m\ddot{x}cos\theta + ml\ddot{\theta} = mgsin\theta \qquad (7)$$

Equations (4) & (7) are the defining equations for this system. These two equations represent a nonlinear system which is relatively complicated from a mathematical view point. However, since the goal of this particular system is to keep the inverted pendulum in upright position around θ =0, the linearization might be considered about this upright equilibrium point.

B. Nonlinear system equations of Inverted pendulum

Nonlinear equations (4) and (7) can be written into standard state space form as:

$$\frac{dx}{dt} = f(\boldsymbol{x}, u, t) \qquad (8)$$

To put equations (4) and (7) into this form, firstly these equations are manipulated algebraically to have only a single second derivative term in each equation. From equation (4) and (8) , we have

$$(M + m - mcos^2\theta)\ddot{x} = u + mlsin\theta\dot{\theta}^2 - mgcos\theta sin\theta \qquad (9)$$

$$\text{and } (mlcos^2\theta - (M+m)l)\ddot{\theta} = ucos\theta - (M+m)gsin\theta + mlcos\theta sin\theta\dot{\theta}^2 \qquad (10)$$

Now these equations may be represented into state space form by considering the state variables as following:

$$x_1 = \theta, \quad x_2 = \dot{\theta} = \dot{x}_1, x_3 = x, x_4 = \dot{x} = \dot{x}_3 \qquad (11)$$

Then, the final state space equation for the inverted pendulum system may be written as:

$$\frac{d}{dt}\boldsymbol{x} = \frac{\boldsymbol{d}}{\boldsymbol{dt}}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{\boldsymbol{d}}{\boldsymbol{dt}}\begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \qquad (12)$$

If both the pendulum angle $\theta$and the cart position x are the variables of interest, then the output equation may be written as: y = **C**x Or, y = $\begin{bmatrix} \theta \\ x \end{bmatrix}$ = **C**x =

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} \qquad (13)$$

Equations (12) and (13) give a complete state space representation of the nonlinear inverted pendulum-cart dynamic system.

C. Linear System Equations of Inverted Pendulum

The linear model for the system around the upright stationary point is derived by simply linearization of the nonlinear system given in equation (12). The linearized form for the system becomes:

$$\frac{d}{dt}\delta\boldsymbol{x} = \boldsymbol{J_x}(\boldsymbol{x_0}\,, u_0)\delta\boldsymbol{x} + \boldsymbol{J_u}(\boldsymbol{x_0}, u_0)\boldsymbol{\delta}\text{u} \qquad (14)$$

Where, the reference state is defined with the pendulum stationary and upright with no input force. Under these conditions, $\boldsymbol{x}_0 = 0, and u_0 = 0$. The componentsof the Jacobian matrices are determined systematically, term by term. The elements of the first, second, third and fourth columns of $\boldsymbol{J_x}(\boldsymbol{x_0}\,, u_0)$ are given by:

The value of $\frac{\partial f_1}{\partial x_1}$, $\frac{\partial f_1}{\partial x_2}$, $\frac{\partial f_1}{\partial x_3} and \frac{\partial f_1}{\partial x_4}$ at $(\boldsymbol{x_0}, \; u_0)$ respectively.

$$\boldsymbol{J_x}(\boldsymbol{x_0}, u_0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)g}{Ml} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-mg}{M} & 0 & 0 & 0 \end{bmatrix} \qquad (15)$$

For the derivative of nonlinear terms with respect to u , we have

$$\boldsymbol{J_u}(\boldsymbol{x_0}, u_0) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_2} \\ \frac{\partial f_3}{\partial u_3} \\ \frac{\partial f_4}{\partial u_4} \end{bmatrix} \quad \text{at } (\boldsymbol{x_0}, \; u_0)$$

Or, $\boldsymbol{J_u}(\boldsymbol{x_0}, u_0)=\begin{bmatrix} 0 \\ \frac{\cos x_1}{ml\cos^2 x_1} \\ 0 \\ \frac{1}{M+m-m\cos^2 x_1} \end{bmatrix}$ at $(\boldsymbol{x_0},\ u_0)=\begin{bmatrix} 0 \\ \frac{-1}{Ml} \\ 0 \\ \frac{1}{M} \end{bmatrix}$ (16)

Finally after all these calculations the equations (14) may be written as :

$$\frac{d}{dt}\boldsymbol{\delta x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)g}{Ml} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-mg}{M} & 0 & 0 & 0 \end{bmatrix}\delta x + \begin{bmatrix} 0 \\ \frac{-1}{Ml} \\ 0 \\ \frac{1}{M} \end{bmatrix}\delta u \quad (17)$$

The above equation (17) represents the linear time invariant system in standard state space form. This may be written in general form as :

$$\frac{d}{dt}\boldsymbol{\delta x} = \mathbf{A}\boldsymbol{\delta x} + \mathbf{B}\delta u \quad (18)$$

The above equation (18) together with the output equation (13) represents the final linear model of the inverted pendulum-cart dynamical system. This is the simplified model which is used to study the system behaviour and controller design.

III. Controller Design

To stabilize the inverted pendulum in upright position and to control the cart at desired position it is necessary to designed some useful controllers which can be used in different control methods applied on system. Some control methods are given as:

A. PID Control

To stabilize the inverted pendulum in upright position and to control the cart at a desired position using PID control approach two PID controllers named angle PID controller and cart PID are implemented. Controllers have been designed for the two control loops of the system. The equations of PID control are given as follow:

$$u_p = K_{pp}e_\theta(t) + K_{ip}\int e_\theta(t) + K_{dp}\frac{de_\theta(t)}{dt} \quad (19)$$

$$u_c = K_{pc}e_x(t) + K_{ic}\int e_x(t) + K_{dc}\frac{de_x(t)}{dt} \quad (20)$$

Where $e_\theta(t)$ and $e_x(t)$ are angleerror andcartposition error. Since the pendulum angle dynamics and cart position dynamics arecoupledtoeach other so the change in any controller parametersaffects both the pendulum angleand cart position which makes the tuning tedious. The tuning of controller parameter is done and observing the responses of SIMULINK model to be optimal.

B. Optimal control using LQR

This technique uses a state -space approach to analyze a system. The method provides a systematic way of computing the state feedback control gain matrix. To determine the matrix K of the optimal control vector, U(t) = -K x(t) and to minimize the performance index.

$$J = \int_0^\infty (x * Qx + u * Ru)dt \quad (21)$$

Where Q is a positive semi-definite and R is a positive definite matrix. Matrices Q and R determine the relative importance of the error.

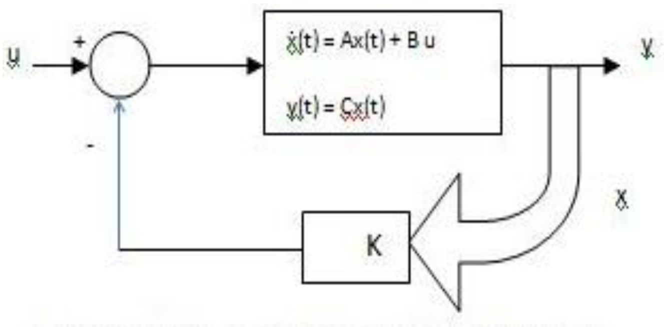

Figure 2: Block diagram for optimal configuration

The equation can be simplified to

$$A * P + PA - PBR^{-1}B * P + Q = 0 \quad (22)$$

Above equation is called Algebraic Riccati Equation (ARE). The ARE can be solved for finding the value of P.

Substituting the matrix P in equation

$$K = R^{-1}B * P \quad (23)$$

The resulting matrix K is the optimal matrix. Using LQR function, two parameters i.e. R and Q can be chosen.R and Q will balance the relative importance of the input. In the Q matrix, the elements in the diagonal matrix represent the weights of state variable.

IV. Simulation and Results

The Matlab-Simulink models for analysis and control of nonlinear inverted pendulum system have been developed. The typical parameters of IP cart system setup are taken as [5-6] mass of the cart (*M*): 2.4 kg, mass of the pendulum (*m*): 0.23 kg, length of the pendulum (*l*): 0.36m, length of the cart track (*L*): ± 0.5 m, friction coefficient of the cart & pole rotation is assumed negligible.

The K-matrix also called LQR gain matrix can be obtained by choosing a suitable value of Q and R using matlab command. For a particular gain matrix the response of IP angle and position can be plotted. Q and R matrix is adjusted by hit and trial method to obtain the desired response. For Q=diag (1,1,10,10) and R=0.1, the value for LQR controller is obtained as K=[-3.16 -6.85 -162.07 -36.92].The simulated results for output variables for inverted pendulum is plotted separately.

In this paper different control methods have been implemented for optimal control of nonlinear IP cart dynamical system. The methods discussed in this paper are a double Proportional-Integral-Derivative (PID) control method, a modern Linear Quadratic Regulator (LQR) control method and a combination of PID and Linear Quadratic Regulator (LQR) control methods. The tuned PID controller parameters of these control schemes are given in table 1.The tuning of PID controller parameters is done using trial & error method and observing the responses of SIMULINK model to be optimal.

TABLE 1: PID controller parameters

Control Schemes	**Angle PID Control**			**Cart PID Control**		
	Kp	**Ki**	**Kd**	**Kp**	**Ki**	**Kd**
2 PID	-40	7	-6	-1	0	-3
2 PID+LQR	1	0.005	8	1.5	-7.5	5
1 PID+LQR	--	--	--	1.5	-6	4

Case 1: *Simulink model for two PID control*

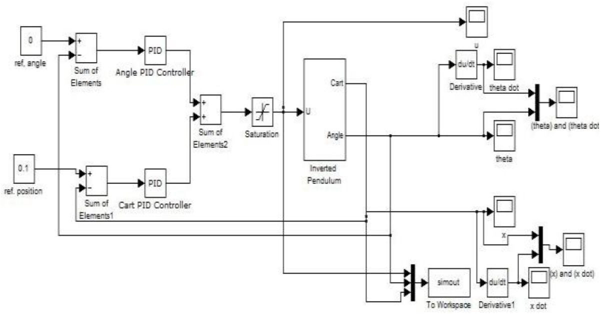

Figure 3: Simulink model for two PID controller

Output plots: i) plot of x(m) and $\dot{x}(m/sec)$ versus time(sec)

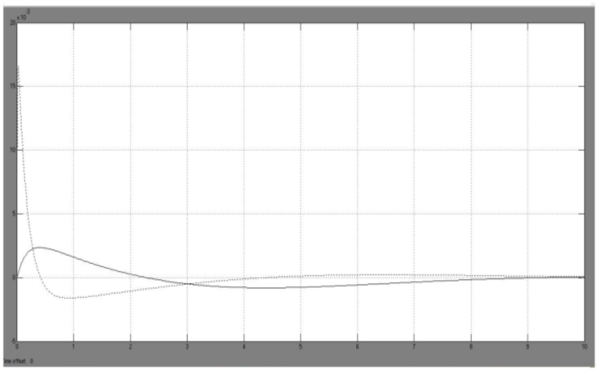

Figure 4: plot of x(m) and $\dot{x}(m/sec)$ versus time(sec)

(ii) Plot of $\theta(rad)\ and\dot{\theta}(rad/sec)$ versus time(sec)

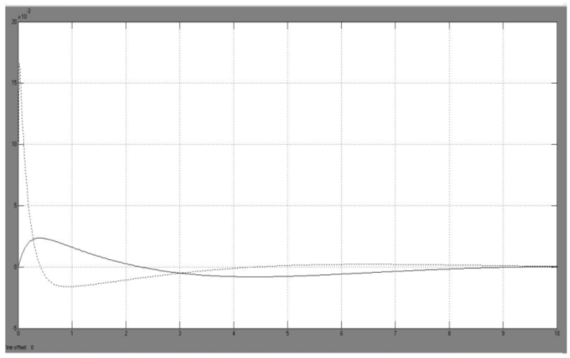

Figure 5: Plot of $\theta(rad)\ and\dot{\theta}(rad/sec)$ versus time

Case 2: *Simulation Result of an Inverted pendulum-cart dynamical system using (LQR) Controller*

For LQR controller as Q=diag (1,1,10,10) and R=0.1, the value for LQR controller is obtained as K=[-3.16 -6.85 -162.07 -36.92]. The simulated results of output variables x(m) , $\dot{x}(m/sec)$ AND $\theta(rad), \dot{\theta}(rad/sec)$ versus time(sec)for inverted pendulum is shown in figure 6.

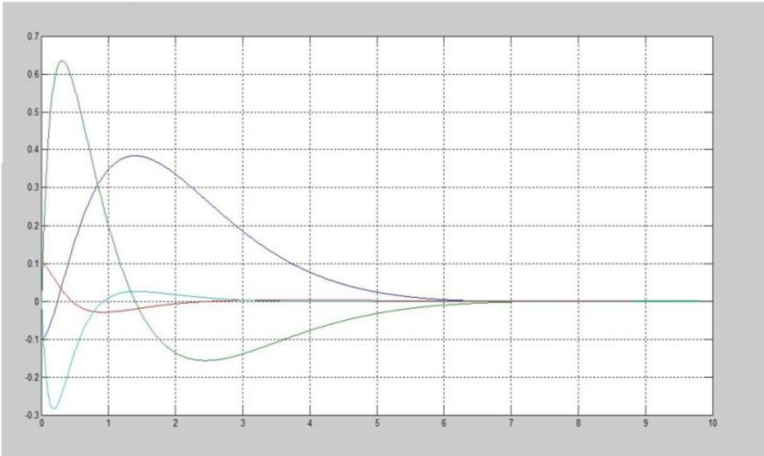

Figure 6: plot of output using LQR controller

Case 3: *Simulink model and Result of an Inverted pendulum-cart dynamical system using (Cart PID + LQR) Controller*

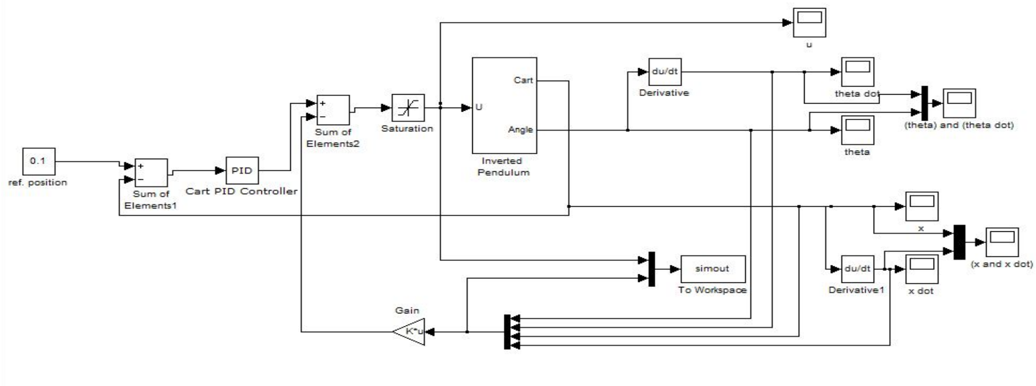

Figure 7: Simulink model of an IP-cart dynamical system using (Cart PID + LQR) Controller

(i) Plot of x(m) and $\dot{x}(m/sec)$ versus time

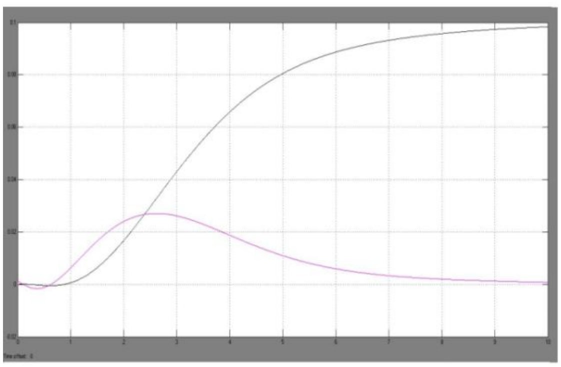

Figure 8: Plot of x(m) and $\dot{x}(m/sec)$ versus time(sec)

(ii) Plot of $\theta(rad) and\ \dot{\theta}$ versus time

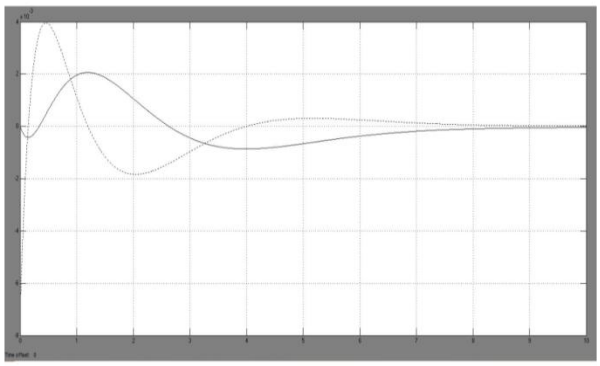

Figure 9: Plot of $\theta(rad)$ $and\dot{\theta}(radsec)$versus time(sec)

Case 4*: Simulink model and Result of an IP-cart dynamical system using (Angle PID, Cart PID +LQR) Controller*

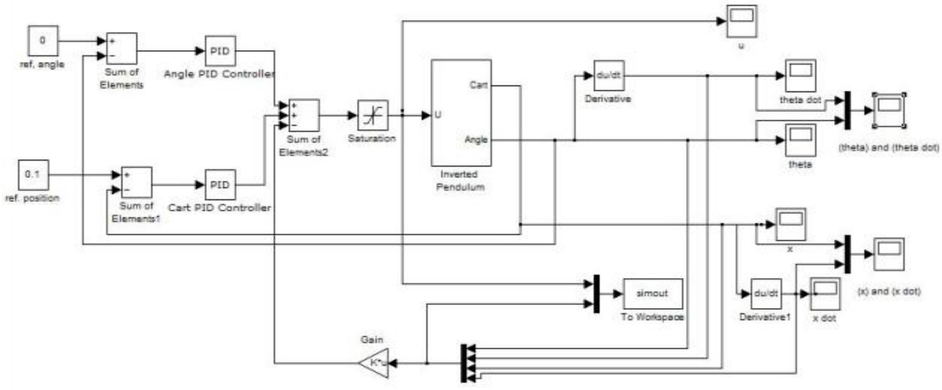

Figure10: Simulink model for IP-cart driven dynamical system using(Angle PID , Cart PID + LQR)Controller

(i)Plot of x(m) and $\dot{x}(m/sec)$ versus time(sec)

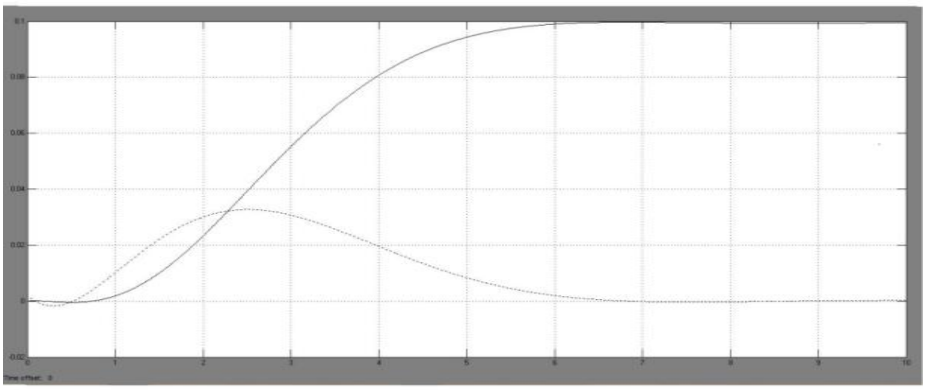

Figure 11: Plot of x(m) and $\dot{x}(m/sec)$ versus time(sec)

(ii) Plot of $\theta(rad)$ $and\dot{\theta}(rad/sec)$ versus time(sec)

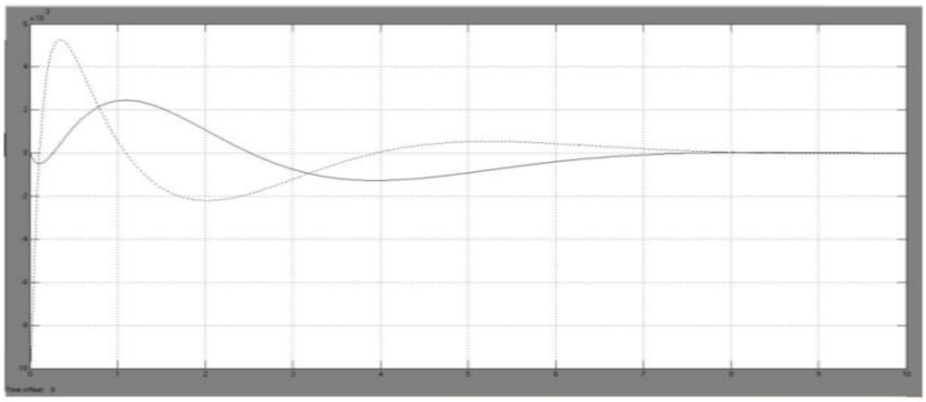

Figure 12: Plot of $\theta(rad)$ $and\dot{\theta}(rad/sec)$ versus time

V. CONCLUSIONS

In this paper PID and LQR controller which are an optimal control technique has been used to make the optimal control decision by using the state-space approach to analyze the system. In the optimal control of nonlinear inverted pendulum dynamical system using PID controller & LQR approach, all the instantaneous states of the nonlinear system are considered available for measurement, which are directly fed to the LQR. The LQR is designed using the linear state space model of the system. The optimal control value of LQR is added negatively with PID control value to have a resultant optimal control.

Simulation results show the comparative advantages of optimal control using LQR method. The pendulum stabilizing in upright position justify that the control schemes are effective and robust. The performance response of the PID controller using LQR is better than PID controller in controlling the inverted pendulum system.Comparing the results it is observed that the responses of both alternatives of PID+LQR control method are better than PID control and LQR control as in LQR control there is severe overshoot and undershoot. It is also observed that the responses of 2PID+LQR control and cart PID+LQR control are nearly similar. Since 2PID+LQR method has additional degree of freedom of control added by the angle PID controller, this will have overall better response under disturbance input. But the cart PID+LQR control has structural simplicity in its credit.

Further improvement need to be done for both the controllers, LQR controller can be improved so that the percentages overshoot for the pendulum's angle does not have very high range whereas PID controller can be improved so that it's settling time might be reduced as faster as LQR controller.

.

REFERENCES

[1] K. Ogata, Modern Control Engineering, 4th ed, Pearson Education (Singapore) Pvt. Ltd., New Delhi, 2005, chapter 12.

[2] K. Ogata, System Dynamics, 4th ed, Pearson Education (Singapore) Pvt. Ltd., New Delhi, 2004.

[3] Ajit K. Mandal, Introduction to Control Engineering, New Age International Pub., New Delhi, 2000, Chapter 13.

[4] G. Ray, S. K. Das, and B. Tyagi, "*Stabilization of Inverted Pendulum via Fuzzy Control*", IE(I) Journal-EL, vol. 88, Sept. 2007, pp. 58-62.

[5] Debasish Chetergee, Amit Patra, and Harish K. Joglekar, "*Swing-up and stabilization of a cart-pendulum system under restricted cart track length*", Elsevier Journal: Systems & Control Letters, vol. 47, 2002, pp. 355 -364.

[6] Yanmei Liu, Zhen Chen, DingyunXue, and XinheXu, "*Real-Time Controlling of Inverted Pendulum by Fuzzy Logic*", Proceedings of the IEEE International Conference on Automation and Logistics, Shenyang, China, August 2009, pp. 1180-1183.

[7] T.J. Koo, "*Stable model reference adaptive fuzzy control of a class of nonlinear systems,*" IEEE Transactions on Fuzzy Systems, vol.9, no.4, pp.624-636, Aug 2001.

[8] K. J. Astrom, and K. Furuta, "*Swinging up a pendulum by energy control",* Elsevier-Pergamon Journal: Automatica, vol. 36, 2000, pp. 287-295.

[9] C. Yong-Yan, L. Zongli, "*Robust stability analysis and fuzzy-scheduling control for nonlinear systems subject to actuator saturation,*" IEEE Transactions on Fuzzy Systems, vol.11, no.1, pp. 57- 67, Feb 2003.